FUNDAMENTAL GROUPS OF RATIONALLY CONNECTED VARIETIES

JÁNOS KOLLÁR

1. Introduction

Let X be a smooth, projective, unirational variety and $U \subset X$ an open set. The aim of this paper is to find a smooth rational curve $C \subset X$ such that the fundamental group of $C \cap U$ surjects onto the fundamental group of U. Following the methods of [Kollár99] and [Colliot-Thélène00], a positive answer over $\mathbb{C}$ translates to a positive answer over any p-adic field. This gives a rather geometric proof of the theorem of [Harbater87] about the existence of Galois covers of the line over p-adic fields (1.4). We also obtain a slight generalization of the results of [Colliot-Thélène00] about the existence of certain torsors over open subsets of the line over p-adic fields (1.6).

If $U = X$ then $\pi_1(X)$ is trivial (cf. (2.3)), thus any rational curve C will do. If $X \setminus U$ is a divisor with normal crossings and C intersects every irreducible component of $X \setminus U$ transversally, then the *normal* subgroup of $\pi_1(U)$ generated by the image of $\pi_1(C \cap U)$ equals $\pi_1(U)$ by a simple argument. (See, for instance, the beginning of (4.2).) It is also not hard to produce rational curves C such that the image of $\pi_1(C \cap U)$ has finite index in $\pi_1(U)$ (cf. (3.3)). These results suggest that we are very close to a complete answer, but surjectivity is not obvious. Differences between surjectivity and finiteness of the index appear in many similar situation; see, for instance, [Kollár95, Part I] or [Napier-Ramachandran98].

The present proof relies on the machinery of rationally connected varieties developed in the papers [Ko-Mi-Mo92a, Ko-Mi-Mo92b, Ko-Mi-Mo92c]. The relevant facts are recalled in Section 2.

The main geometric result is the following:

Theorem 1.1. *Let K be an algebraically closed field of characteristic zero and X a smooth, projective variety over K which is rationally connected (2.1). Let $U \subset X$ be an open subset and $x_0 \in U$ a point. Then there is an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$*

such that $f(0) = x_0$ and

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \textit{is surjective.}$$

Moreover, we can assume that the following also hold:

1. $H^1(\mathbb{P}^1, \bar{f}^* T_X(-2)) = 0$ *where* $\bar{f} : \mathbb{P}^1 \to X$ *is the unique extension of* f.
2. $\bar{f}$ *is an embedding if* $\dim X \geq 3$ *and an immersion if* $\dim X = 2$.

Corollary 1.2. *Let K be a p-adic field and X a smooth, projective variety over K which is rationally connected over $\bar{K}$. Let $U \subset X$ be an open subset and $x_0 \in U(K)$ a point. Then there is an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$ (all defined over K) such that $f(0) = x_0$ and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \textit{is surjective,}$$

where π_1 here denotes the algebraic fundamental group.

Remark 1.3. More generally, (1.2) holds for any field K of characteristic zero such that every curve with a smooth K-point contains a Zariski dense set of K-points. Characterizations of this property are given in [Pop96, 1.1]. The following are some interesting classes of such fields:

1. Fields complete with respect to a discrete valuation.
2. Quotient fields of local Henselian domains.
3. $\mathbb{R}$ and all real closed fields.
4. Pseudo algebraically closed fields, cf. [Fried-Jarden86, Chap. 10]

Corollary 1.4. [Harbater87] *Let G be a finite group and K a field of characteristic zero as in (1.3). Then there is a Galois cover $g : C \to \mathbb{P}^1_K$ with Galois group G such that C is geometrically irreducible and $g^{-1}(0{:}1) \cong G$.*

Proof. Let $G \subset GL(n, K)$ be a faithful representation. Set $U = GL(n)/G$ with quotient map $h : GL(n) \to U$ and let x_0 be the image of the identity matrix. Then U is unirational, thus by (1.2) there is a $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$ such that $\pi_1(V) \twoheadrightarrow \pi_1(U)$ is onto. $h : GL(n) \to U$ is étale and proper, thus it corresponds to a quotient $\pi_1(U) \twoheadrightarrow G$. The fiber product $W := GL(n) \times_U V \to V$ corresponds to the surjective homomorphism

$$\pi_1(V) \twoheadrightarrow \pi_1(U) \twoheadrightarrow G.$$

Thus W is connected and $W \to V$ is a Galois cover with Galois group G. Since W has a K-point, it is also geometrically connected. The

preimage of $0 \in V$ is isomorphic to G (the disjoint union of $|G|$ copies of Spec_K). $W \to V$ can be extended to a (ramified) Galois cover of the whole $\mathbb{P}^1_K$. □

Remark 1.5. The above proof works in positive characteristic if we know that for every subgroup $H < G$ the quotient $GL(n)/H$ has a smooth compactification.

The following result was proved by [Colliot-Thélène00] for finite groups, which is probably the most important for applications.

Corollary 1.6. *Let K be a field of characteristic zero as in (1.3), G a linear algebraic group scheme over K and A a principal homogeneous G-space. Then there is an open set $0 \in V \subset \mathbb{A}^1_K$ and a geometrically irreducible G-torsor $g : W \to V$ such that $g^{-1}(0) \cong A$ (as a G-space).*

Proof. Assume that G acts on A from the left and choose an embedding $G \subset GL(n)$ over K. $A \times GL(n)$ has a diagonal left action by G and a right action by $GL(n)$ acting only on $GL(n)$.

The right G-action makes the morphism $h : G\backslash(A \times GL(n)) \to G\backslash(A \times GL(n))/G =: U$ into a G-torsor. Let $x_0 \in U$ be the image of $G\backslash(A \times G)$. The fiber of h over x_0 is isomorphic to A. Let G^0 be the connected component of G. Then $G\backslash(A \times GL(n)) \to G\backslash(A \times GL(n))/G^0$ is smooth with connected fibers and $G\backslash(A \times GL(n))/G^0 \to U$ is étale and proper. Let $0 \in V \subset \mathbb{A}^1$ and $f : V \to U$ be as in the proof of (1.4). Then $W := (G\backslash(A \times GL(n))) \times_U V$ works. □

A geometric application is the following:

Corollary 1.7. *For every $2 \leq g \leq 13$ there is an open set $0 \in V_g \subset \mathbb{C}$ and a smooth proper morphism with genus g fibers $S_g \to V_g$ such that the image of the monodromy representation is the full Teichmüller group.*

Proof. The moduli of curves is unirational for $g \leq 13$. Apply (1.1) to the open subset of curves without automorphisms $U_g \subset M_g$. □

2. Rationally connected varieties and morphisms of rational curves

Rationally connected varieties were introduced in [Ko-Mi-Mo92b] as a higher dimensional generalization of rational and unirational varieties. A surface is rationally connected iff it is rational. In higher dimensions rationality and unirationality are very hard to check. The notion of rational connectedness concentrates on rational curves on a variety. The following characterizations were developed in the papers [Ko-Mi-Mo92b], [Kollár96, IV.3], [Kollár98, 4.1.2].

Definition–Theorem 2.1. Let K be an algebraically closed field of characteristic zero. A smooth proper variety X over K is called *rationally connected* if it satisfies any of the following equivalent properties:

1. There is an open subset $\emptyset \neq X^0 \subset X$, such that for every $x_1, x_2 \in X^0$, there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{P}^1)$.
2. For every $x_1, \dots, x_n \in X$, there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, \dots, x_n \in f(\mathbb{P}^1)$.
3. There is a morphism $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. (This is equivalent to f^*T_X being ample.)
4. There is a variety P and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that $F((0\!:\!1) \times P)$ is a point. We can also assume that $H^1(\mathbb{P}^1, F_p^*T_X(-2)) = 0$ for every p where $F_p := F|_{\mathbb{P}^1 \times \{p\}}$.
5. Let $z_1, \dots, z_n \in \mathbb{P}^1$ be distinct points and $m_1, \dots, m_n$ natural numbers. For each $i = 1, \dots, n$ let $f_i : \operatorname{Spec} K[t]/(t^{m_i}) \to X$ be a morphism. Then there is a variety P and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that
   (a) the Taylor series of F_p at z_i coincides with f_i up to order m_i for every i and $p \in P$,
   (b) $H^1(\mathbb{P}^1, F_p^*T_X(-\sum m_i)) = 0$ for every i and p. □

Another easy result that we need is the following.

Lemma 2.2. *(cf.* [Kollár96, II.3.5.4,II.3.10.1 and II.3.11]*) Let X, P be smooth varieties and $F : \mathbb{P}^1 \times P \to X$ a dominant morphism such that $F((0\!:\!1) \times P)$ is a point. Then there is a dense open set $P^0 \subset P$ such that $H^1(\mathbb{P}^1, F_p^*T_X(-2)) = 0$ for every $p \in P^0$ where $F_p := F|_{\mathbb{P}^1 \times \{p\}}$.*

Conversely, let $f : \mathbb{P}^1 \to X$ be a morphism such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. Then there is a pointed variety $p_0 \in P$ and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that $F((0\!:\!1) \times P)$ is a point, F is smooth away from $(0\!:\!1) \times P$ and $F_{p_0} = f$. □

The following result was proved by [Serre59] for unirational varieties and by [Campana91, Ko-Mi-Mo92c] in general.

Proposition 2.3. *A smooth, proper, rationally connected variety is simply connected.* □

In the course of the proof we repeatedly encounter the following situation. We have morphisms $f^i : \mathbb{P}^1 \to X$ each passing through the same point $x_0 \in X$. We would like to have a family of morphisms $f_t : \mathbb{P}^1 \to X$ such that the union of the maps f^i can be considered as the limit of the maps f_t as $t \mapsto 0$. The following lemma is a technical formulation of this idea. Its statement is a bit complicated since we also want to keep track of the field over which the f_t are defined.

Lemma 2.4. *(cf. [Kollár99, 3.2]) Let K be a field and $x_0 \in X$ a smooth, proper, pointed K-scheme. Let S be a zero dimensional reduced K-scheme and $f_0 : \mathbb{P}^1_S \to X$ a morphism such that*

1. $H^1(\mathbb{P}^1_S, f_0^* T_X(-2)) = 0$, *and*
2. $f_0(S \times \{(0:1)\}) = \{x_0\}$.

Then there are

3. *a smooth pointed curve $0 \in D$ over K,*
4. *a smooth surface Y with a proper morphism $h : Y \to D$ and a section $B \subset Y$ of h, and*
5. *a morphism $F : Y \to X$,*

such that

6. *$h^{-1}(0)$ is the union of $\mathbb{P}^1_S$ with a copy B_0 of $\mathbb{P}^1_K$ such that $B_0 \cap \mathbb{P}^1_S = S \times \{(0:1)\}$ and $B_0 \cap B$ is a single point.*
7. *F restricted to $\mathbb{P}^1_S$ coincides with f_0 and $F(B_0 \cup B) = \{x_0\}$,*
8. *$h^{-1}(D^0) \cong \mathbb{P}^1_K \times D^0$, where $D^0 := D \setminus \{0\}$,*
9. *$H^1(\mathbb{P}^1_d, F_d^* T_X(-2)) = 0$ for every $d \in D^0$ where $F_d := F|_{h^{-1}(d)}$.*

Proof. Let us start with any curve $0 \in D'$ and $Y' := \mathbb{P}^1 \times D'$. Let $S' \subset \mathbb{P}^1 \times \{0\}$ be a subscheme isomorphic to S and Y'' the blow up of $S' \subset D'$ with projection $h' : Y'' \to D'$. We can define a morphism $f' : (h')^{-1}(0) \to X$ by setting f' to be f_0 on the exceptional divisor of $Y'' \to Y'$ and the constant morphism to $\{x_0\}$ on the birational transform of $\mathbb{P}^1 \times \{0\}$. Fix any section $B' \subset Y'$ which does not pass through S'. We are done if f' can be extended to $F' : Y'' \to X$ as required. In general this is not possible, but such an extension exists after a suitable étale base change $(0 \in D) \to (0 \in D')$. This is proved in [Ko-Mi-Mo92b, 1.2] and [Kollár99, 2.2]. □

3. Fundamental groups of fibers of morphisms

We need some easy results about the variation of fundamental groups for fibers of nonproper morphisms.

Lemma 3.1. *Let K be an algebraically closed field of characteristic zero, $\bar{Z}, D$ irreducible K-varieties and $f : \bar{Z} \to D$ a smooth and proper morphism with connected fibers. Let $Z \subset \bar{Z}$ be an open subset such that $(\bar{Z} \setminus Z) \to D$ is smooth. Let $z_0 \in Z(K)$ be a K-point, $d_0 = f(z_0)$ and Z_0 the fiber of $Z \to D$ through z_0. Then there is an exact sequence*

$$\pi_1(Z_0, z_0) \to \pi_1(Z, z_0) \to \pi_1(D, d_0) \to 1.$$

Proof. Over $\mathbb{C}$ the fibration $Z(\mathbb{C}) \to D(\mathbb{C})$ is a topological fiber bundle, thus we have the above exact sequence. To settle the algebraic case, let $Y \to Z$ be any connected finite degree étale cover and extend

it to a finite morphism $\bar{Y} \to \bar{Z}$ where $\bar{Y}$ is normal. Since $\bar{Z} \setminus Z$ is smooth over D, the same holds for $\bar{Y} \to D$. (This is a special case of Abhyankar's lemma, cf. [SGA1, XIII.5.2].) The generic fiber of $\bar{Y} \to D$ is irreducible. Since $\bar{Y} \to D$ is smooth and proper, every fiber is irreducible. This is equivalent to the exactness of the above sequence. □

The following technical lemma is an upper semi-continuity statement for the fundamental groups of fibers of nonproper morphisms.

Lemma 3.2. *Let K be an algebraically closed field of characteristic zero, W a normal surface over K, $f : W \to D$ a (not necessarily proper) morphism to a curve with connected fibers. Let $B \subset W$ be a connected subset, one of whose irreducible components is a section of f. Let $d_0 \in D$ be a K-point and C_0 an irreducible component of $f^{-1}(d_0)$ with a K-point $b_0 \in C_0 \cap B$. Let $x_0 \in U$ be a pointed K-scheme and $h : W \to U$ a morphism such that $h(B) = \{x_0\}$. Then there is an open subset $D^0 \subset D$ such that for every $d \in D^0$ and for every K-point $b_d \in C_d \cap B$,*

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(C_0, b_0) \to \pi_1(U, x_0)].$$

Proof. Choose a normal compactification $\bar{f} : \bar{W} \to D$. Let $D^0 \subset D$ be any open subset such that $\bar{f}$ is smooth with irreducible fibers over D^0 and $\bar{W} \setminus W \to D$ is unramified over D^0. Set $W^0 := f^{-1}(D^0)$. $\pi_1(C_0, b_0) \to \pi_1(U, x_0)$ factors through $\pi_1(W, b_0) \to \pi_1(U, x_0)$ and it has the same image as $\pi_1(W, b) \to \pi_1(U, x_0)$ for any $b \in B(K)$. $\pi_1(W^0, b) \to \pi_1(W, b)$ is surjective by (3.3). By (3.1) there is an exact sequence

$$\pi_1(C_d, b_d) \to \pi_1(W_0, b_d) \to \pi_1(D^0, d) \to 1.$$

Let $B^0 \subset B \cap W^0$ be a section of f. Then $\pi_1(B^0, b_d)$ maps onto $\pi_1(D^0, d)$ and the image of $\pi_1(B^0, b_d)$ in $\pi_1(U, x_0)$ is trivial. Thus

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] = \operatorname{im}[\pi_1(W^0, b_d) \to \pi_1(U, x_0)]$$

and we are done. □

Lemma 3.3. *(cf.* [SGA1, IX.5.6], [Campana91, 1.3]*). Let X, Y be normal varieties, $x_0 \in X$ a closed point and $f : X \to Y$ a dominant morphism. Then $\operatorname{im}[\pi_1(X, x_0) \to \pi_1(Y, f(x_0))]$ has finite index in $\pi_1(Y, f(x_0))$. If f is an open immersion then $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.* □

4. Proof of the main results

The theory of free morphisms of curves (cf. [Kollár96, II.3]) suggests that morphisms $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$ behave rather predictably. Therefore we concentrate on such morphisms. First we establish that there is a unique maximal subgroup of $\pi_1(U)$ obtainable from such a morphism.

Lemma 4.1. *Let X be a smooth, proper, rationally connected variety over an algebraically closed field of characteristic zero. Let $U \subset X$ be an open set and $x_0 \in U$ a point. Then there is a unique finite index subgroup $H < \pi_1(U, x_0)$ with the following properties:*

1. *There is a morphism $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$, $f(0{:}1) = x_0$ and $H = \operatorname{im}[\pi_1(f^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$.*
2. *Let g be any morphism $g : \mathbb{P}^1 \to X$ such that $g(0{:}1) = x_0$ and $H^1(\mathbb{P}^1, g^*T_X(-2)) = 0$. Then $H \supset \operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$.*

Proof. First we find one morphism as in (4.1.2) such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$$

has finite index in $\pi_1(U, x_0)$.

Let $F : \mathbb{P}^1 \times P \to X$ be as in (2.1.4). Let $P^0 \subset P$ be an open subset such that $F^{-1}(X \setminus U) \to P$ is étale over P^0. Pick any poit $p \in P^0$. By (3.1) there is an exact sequence

$$\begin{aligned} \pi_1(F_p^{-1}(U), (0{:}1) \times \{p\})) &\to \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0{:}1) \times \{p\})) \\ &\to \pi_1(P^0, p) \to 1. \end{aligned}$$

The section $(0{:}1) \times P^0$ is mapped to a point by F, thus

$$\pi_1(F_p^{-1}(U), (0{:}1) \times \{p\})) \quad \text{and} \quad \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0{:}1) \times \{p\})$$

have the same image in $\pi_1(U, x_0)$. The latter image has finite index by (3.3), thus $g := F_p : \mathbb{P}^1 \to X$ is as desired.

In order to finish, it is sufficient to prove that if $g_1, g_2 : \mathbb{P}^1 \to X$ are as in (4.1.2) then there is a third morphism $g : \mathbb{P}^1 \to X$ such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(g_i^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$$

for $i = 1, 2$. To do this, let $S = \{1, 2\}$ be a 2 point scheme and take $f_0 : \mathbb{P}^1_S \to X$ to be g_i on $\{i\} \times \mathbb{P}^1$. Construct $h : Y \to D$ and $F : Y \to X$ as in (2.4). Set $W := F^{-1}(U)$ and apply (3.2) twice for $i = 1, 2$ with $C_0 := f_i^{-1}(U)$. For any $d \in D^0$, $g := F_d$ has the required property. □

4.2 (Proof of (1.1))**.**

Let $H < \pi_1(U, x_0)$ be the subgroup obtained in (4.1). We are done if $H = \pi_1(U, x_0)$. Otherwise there is a corresponding irreducible étale

cover $(x_0' \in U') \to (x_0 \in U)$. Let $w : X' \to X$ be the normalization of X in the function field of U'. X is simply connected by (2.3), hence by the purity of branch loci (cf. [SGA1, X.3.1]) there is a divisor $D' \subset X'$ such that w ramifies along D'. Let $D \subset X$ be the image of D'. By construction $D \subset X \setminus U$. We derive a contradiction as follows.

Let $f : \mathbb{P}^1 \to X$ be a morphism such that $f(0{:}1) = x_0$ and $f(\mathbb{P}^1)$ intersects D transversally at a point $x_1 := f(1{:}1)$. This implies that the local fundamental group of $\mathbb{P}^1 \setminus \{(1{:}1)\}$ at $(1{:}1)$ surjects onto the local fundamental group of $X \setminus D$ at x_1. (See [Grothendieck-Murre71] for the local fundamental group of a divisor in a variety. There it is called the fundamental group of the formal neighborhood of a divisor.) Therefore, if f lifts to $f' : \mathbb{P}^1 \to X'$ then $X' \to X$ is étale at $f'(1{:}1)$. If $X' \to X$ is a Galois extension the we already have a contradiction since $X' \to X$ ramifies everywhere above D. In the non–Galois case p may be unramified along some of the irreducible components of $p^{-1}(D)$, and we only get a contradiction if the image of f' intersects D'.

If $X' \to X$ is not Galois, we need to proceed in a somewhat roundabout way. First I give the outlines; a precise version is given afterwards.

It is clear from the definition that there are many maps $\mathbb{P}^1 \to X'$, thus X' (or rather any desingularization of X') should be rationally connected. This indeed follows from (4.3) applied to any $F : \mathbb{P}^1 \times P \to X$ as in (2.1.4). Thus by (2.1.5) there is a morphism $f' : \mathbb{P}^1 \to X'$ which passes through x_0' and intersects D' at a smooth point x_1'. We obtain a contradiction if $w \circ f'$ is the limit of a sequence of maps $f_t : \mathbb{P}^1 \to X$ such that

1. f_t lifts to $f_t' : \mathbb{P}^1 \to X'$ and f' is the limit of the maps f_t',
2. the image of f_t intersects D transversally.

First we prove a general lifting property for families of morphisms and then we proceed to construct the morphism f'.

Lemma 4.3. *Let V be a normal variety and $G : \mathbb{P}^1 \times V \to X$ a morphism such that $G((0{:}1) \times V) = \{x_0\}$. Assume that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for some $v \in V$. Then G can be lifted to $G' : \mathbb{P}^1 \times V \to X'$.*

Proof. First we show that such a lifting exists over an open subset of $\mathbb{P}^1 \times V$. Choose $V^0 \subset V$ such that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for every $v \in V^0$ (this is possible by the upper semi continuity of cohomology groups) and such that we have an exact sequence

$$\pi_1(G_v^{-1}(U), (0{:}1) \times v) \to \pi_1((G^0)^{-1}(U), (0{:}1) \times v) \to \pi_1(V^0, v) \to 1$$

for every $v \in V^0$ where $G^0 := G|_{\mathbb{P}^1 \times V^0}$ (this is possible by (3.1)). Then

$$\operatorname{im}[\pi_1((G^0)^{-1}(U), (0{:}1) \times v) \to \pi_1(U, x_0)] \subset H,$$

thus $G^0|_{(G^0)^{-1}(U)}$ can be lifted to $G' : (G^0)^{-1}(U) \to U'$. This extends to a rational map $G' : \mathbb{P}^1 \times V^0 \dashrightarrow X'$.

Next let $\Gamma \subset (\mathbb{P}^1 \times V) \times X$ be the graph of G and $\Gamma^* \subset (\mathbb{P}^1 \times V) \times X'$ its preimage. The rational lifting G' corresponds to an irreducible component $\Gamma' \subset \Gamma^*$ such that the projection $\Gamma' \to (\mathbb{P}^1 \times V)$ is birational. Since $X' \to X$ is finite, so is $\Gamma^* \to \Gamma$. Thus $\Gamma' \to \Gamma \to \mathbb{P}^1 \times V$ is finite and birational, hence an isomorphism. □

Let $\phi : X'' \to X'$ be any desingularization and $x'_1 \in D'$ a point such that D' and X' are smooth at x'_1 and ϕ^{-1} is a local isomorphism near x'_1. By (2.1.5) there is a dominant morphism $F : \mathbb{P}^1 \times P \to X'$ such that

1. $F((0{:}1) \times P) = \{x'_0\}$,
2. $F((1{:}1) \times P) = \{x'_1\}$, and
3. the image of $F_p : \mathbb{P}^1 \to X'$ is transversal to D' at x'_1 for every $p \in P$.

Let us now consider the dominant morphism $w \circ F : \mathbb{P}^1 \times P \to X$. By the first part of (2.2), there is a $p_0 \in P$ such that $H^1(\mathbb{P}^1, (w \circ F_{p_0})^* T_X(-2)) = 0$. Thus by the second part of (2.2) there is a pointed variety $q_0 \in Q$ and a morphism $G : \mathbb{P}^1 \times Q \to X$ such that $G_{q_0} = w \circ F_{p_0}$ and G is smooth away from $(0{:}1) \times Q$. In particular, $G^{-1}(D) \subset \mathbb{P}^1 \times Q$ is a generically smooth divisor, hence there is a dense open set $Q^0 \subset Q$ such that the projection $G^{-1}(D) \to Q$ is smooth over Q^0. This means that the image of $G_q : \mathbb{P}^1 \to X$ intersects D transversally for every $q \in Q^0$. (Note that $G^{-1}(D)$ denotes the inverse image scheme.)

By (4.3), G can be lifted to $G' : \mathbb{P}^1 \times Q \to X'$. On $\mathbb{P}^1 \times \{q_0\}$ the lifting agrees with F_{p_0}, hence $G'(\mathbb{P}^1 \times \{q_0\})$ intersects D' at the point x'_1. This implies that $(G')^{-1}(D')$ is a divisor and so by shrinking Q we may assume that the image of G'_q intersects D' for every $q \in Q$. Thus $G_q = w \circ G'_q$ is never transversal to D, a contradiction.

Condition (1.1.1) holds by construction and a general choice of f satisfies (1.1.2) by [Kollár96, II.3.14]. □

4.4 (Proof of (1.2))**.**

Pick $f_1 : V_1 \to U_{\bar{K}}$ defined over $\bar{K}$ such that $\pi_1(V_1, 0) \to \pi_1(U_{\bar{K}}, x_0)$ is surjective and $H^1(\mathbb{P}^1, \bar{f}_1^* T_X(-2)) = 0$. f_1 is defined over a finite Galois extension $L \supset K$; let $f_i : V_i \to U_{\bar{K}}$ be its conjugates. Each of these extends to a morphism $\bar{f}_i : \mathbb{P}^1 \to X_{\bar{K}}$. Let $S = \operatorname{Spec}_K L$. We can view the morphisms f_i as one morphism $f_0 : \mathbb{P}^1_S \to X$ defined over K. By (2.4) we obtain $h : Y \to D$ and $F : Y \to X$, all defined over K. Let $d \in D^0(\bar{K})$ be any point. Then by (3.2) we see that

$$\operatorname{im}[\pi_1(Y_d, 0) \xrightarrow{F_d} \pi_1(U_{\bar{K}}, x_0)] \supset \operatorname{im}[\pi_1(V_1, 0) \xrightarrow{f_1} \pi_1(U_{\bar{K}}, x_0)],$$

and the latter image is $\pi_1(U_{\bar K}, x_0)$ by assumption. D^0 is a Zariski open set in a curve D with a smooth K-point 0. By (1.3) this implies that $D(K)$ is dense in D, hence $D^0(K) \neq \emptyset$. By choosing $d \in D^0(K)$ we obtain an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$ (all defined over K) such that $f(0) = x_0$ and

$$\pi_1(V_{\bar K}, 0) \twoheadrightarrow \pi_1(U_{\bar K}, x_0) \quad \text{is surjective.}$$

The fundamental group of a K-scheme W is related to the fundamental group of $W_{\bar K}$ by the exact sequence (cf. [SGA1, IX.6.1])

$$1 \to \pi_1(W_{\bar K}, 0) \to \pi_1(W, 0) \to \mathrm{Gal}(\bar K/K) \to 1.$$

This implies that

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \text{is also surjective.}$$

$\square$

Example 4.5. For every $n \geq 4$ here I give an example of a rational threefold X, a normal crossing divisor $F \subset X$ and a smooth rational curve $B \subset X$ such that B intersects F everywhere transversally, B intersects every irreducible component of F and the image of $\pi_1(B \setminus F) \to \pi_1(X \setminus F)$ has index n in $\pi_1(X \setminus F)$.

Let us start with a similar surface example. Let $g_1 : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree n morphism with critical points $x_1, \dots, x_{2n-2} \in \mathbb{P}^1$ and different critical values $y_1, \dots, y_{2n-2}$, Choose 3 other points $x_{2n-1}, x_{2n}, x_{2n+1} \in \mathbb{P}^1$ such that $g_1(x_{2n-1})$ and $g_1(x_{2n})$ are different critical values of g_1 and $g_1(x_{2n+1})$ is not a critical value of g_1. Let $g_2 : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism with $g_2(x_i) = x_i$ for $i \leq 2n+1$. Consider the morphism $h : (g_2, g_1) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

Set $D = \mathbb{P}^1 \times \{y_1, \dots, y_{2n-2}\}$ and $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$.

$\operatorname{im} h$ intersects every irreducible componnet of D transversally at $n-2$ points and the image of $\pi_1(h^{-1}(U)) \to \pi_1(U)$ has index n in $\pi_1(U)$.

Here $\operatorname{im} h$ is also tangent to every irreducible component of D. The tangencies can be resolved by 2n blow ups, but then the birational transform of $\operatorname{im} h$ does not intersect every boundary component. To remedy this situation, take another morphism $g_3 : \mathbb{P}^1 \to \mathbb{P}^1$ with $g_3(x_i) = x_i$ for $i \leq 2n+1$. Set $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $F = D \times \mathbb{P}^1$ and $H := (g_3, g_2, g_1) : \mathbb{P}^1 \to Y$. Again the only problem is that $\operatorname{im} H$ is tangent to every irreducible component of F. These can be resolved by blowing up two suitable smooth curves. First we take a smooth curve which passes through every point of tangency and also through the point $H(x_{2n-1})$. After blow up, the birational transform of $\operatorname{im} H$ intersects every boundary component transversally but above each point

there is a point common to two boundary components and to the birational transform of $\operatorname{im} H$. Next take a smooth curve which passes through all these points with a general tangent direction there and also through $H(x_{2n})$. We can also assume that neither of the two curves passes through $H(x_{2n+1})$. Doing two such blow ups creates two new boundary components and the birational transform of $\operatorname{im} H$ intersects both of them. The fundamental group computation is unchanged.

Varying g_2, g_3 we obtain many morphisms all of which pass through the point $H(x_{2n+1})$.

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References

Campana91. F. Campana, On twistor spaces of the class C, J. Diff. Geom. 33 (1991) 541-549

Colliot-Thélène00. J.-L. Colliot-Thélène, Rational connectedness and Galois covers of the projective line, Annals of Math. 151 (2000) 359-373

Fried-Jarden86. M. Fried and M. Jarden, Field arithmetic, Springer–Verlag, 1986

Grothendieck-Murre71. A. Grothendieck and J. Murre, The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme, Springer Lecture Notes vol. 208 (1971)

Harbater87. D. Harbater, Galois coverings of the arithmetic line, Number Theory Seminar, New York 1984/85, Springer Lecture Notes vol. 1240, 1987, 165-195,

Hartshorne77. R. Hartshorne, Algebraic Geometry, Springer, 1977

Kollár95. J. Kollár, Shafarevich maps and automorphic forms, Princeton Univ. Press, 1995

Kollár96. J. Kollár, Rational Curves on Algebraic Varieties, Springer Verlag (1996) Ergebnisse der Math. vol. 32

Kollár98. J. Kollár, Low degree polynomial equations, European Congress of Math. Birkhäuser, 1998, 255-288

Kollár99. J. Kollár, Rationally connected varieties over local fields, Annals of Math. 150 (1999) 357–367

Ko-Mi-Mo92a. J. Kollár, Y. Miyaoka and S. Mori, Rational Curves on Fano Varieties, Proc. Alg. Geom. Conf. Trento, Springer Lecture Notes 1515 (1992) 100-105

Ko-Mi-Mo92b. J. Kollár, Y. Miyaoka and S. Mori, Rationally Connected Varieties, J. Alg. Geom. 1 (1992) 429-448

Ko-Mi-Mo92c. J. Kollár, Y. Miyaoka and S. Mori, Rational Connectedness and Boundedness of Fano Manifolds, J. Diff. Geom. 36 (1992) 765-769

Napier-Ramachandran98. T. Napier and M. Ramachandran, The L^2 $\overline{\partial}$-method, weak Lefschetz theorems, and the topology of Kähler manifolds, J. Amer. Math. Soc. 11 (1998) 375–396

Pop96. F. Pop, Embedding problems over large fields, Annals of Math, 144 (1996) 1-34

Serre59. J.-P. Serre, On the fundamental group of a unirational variety, J. London Math. Soc. 34 (1959) 481–484

SGA1. A. Grothendieck, Revêtements étales et groupes fondamental, Springer Lecture Notes vol. 224, 1971

Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu